SOME PROBABILISTIC THEOREMS ON DIOPHANTINE APPROXIMATIONS[1]

BY

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1. Introduction. Let

$$\langle \xi \rangle = \min_{k \text{ integer}} |\xi - k|$$

be the positive distance between ξ and the nearest integer to ξ. The first theorem is concerned with

$$\min_{1 \leqq k \leqq m} \langle k\xi \rangle.$$

By the methods used for Theorem 1 we also solve some special cases of a problem raised in [2] concerning the existence of an integer k for which $m \leqq k \leqq mc$ $(c > 1)$ and $\langle k\xi \rangle \leqq \alpha/k$ (Theorem 2).

While studying $\min_{1 \leq k \leq n} \langle k\xi \rangle$, one is naturally led to consider the number of integers k for which $1 \leqq k \leqq m$ and $\langle k\xi \rangle \leqq \gamma$. The third theorem deals with this quantity in p dimensions i.e. it considers $N(m, \gamma, p)$, the number of integers k, for which $1 \leqq k \leqq m$ and *simultaneously*

$$\langle k\xi_1 \rangle \leqq \gamma, \ \langle k\xi_2 \rangle \leqq \gamma, \cdots, \langle k\xi_p \rangle \leqq \gamma. \tag{1.1}$$

Theorem 4 gives some easy generalizations of the third theorem.

Our approach is probabilistic in the sense that we do not take $\xi, \xi_1, \cdots, \xi_p$ fixed, but choose them randomly, according to a uniform distribution on $[0, 1]$. This makes $\min_{1 \leq k \leq m} \langle k\xi \rangle$ and $N(m, \gamma, p)$ random variables. Accordingly, Theorem 1 gives an asymptotic expression for the Lebesgue measure of the set

$$\left\{ \xi : 0 \leqq \xi \leqq 1, \ m \cdot \min_{1 \leqq k \leqq m} \langle k\xi \rangle \leqq \alpha \right\}.$$

Theorem 3, which states that $N(m, \gamma, p)$ has asymptotically $(p \to \infty, \ m \cdot (2\gamma)^p \to \lambda)$ a Poisson distribution with mean λ, can be paraphrased similarly. It then gives asymptotic expressions for the p dimensional Lebesgue measure of the sets

$$\{\xi_1, \cdots, \xi_p : 0 \leqq \xi_j \leqq p, \ N(m, \gamma, p) = k\}, \qquad k = 0, 1, \cdots.$$

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We shall stick further to the probabilistic language. As far as used here it is of such a simple kind that it should not cause any difficulties. All the required definitions can be found in [9].

Theorems 1, 2 are immediate extensions of the results of Friedman and Niven [3] and of Erdös, Szüsz, and Turan [2]. Theorem 3 is proved by the method of moments. We are forced, however, to prove the convergence of the moments in a rather roundabout way (cf. also the remarks after Lemma 1). One should compare Theorem 3 with the well-known result of Dirichlet, which states that if γ^{-1} is an integer, then there exists for *every* $(\xi_1, \cdots, \xi_p)$ at least one $k \leqq \gamma^{-p}$ satisfying (1.1). Our results show that the Lebesgue measure of the set of $(\xi_1, \cdots, \xi_p)$'s, for which there exists such a $k \leqq \lambda(2\gamma)^{-p}$, is approximately $1-e^{-\lambda}$. Related results were also obtained by Erdös and Rényi [1] who studied

$$\max_{1 \leqq k \leqq m} \left| \sum_{j=1}^{p} e^{2\pi i k \xi_j} \right|.$$

2. **The minimum of** $\langle kx \rangle$, $k=1, \cdots, m$. Let x have a uniform distribution(2) on $[0, 1]$ and put

$$Z(m) = \min_{1 \leqq k \leqq m} \langle kx \rangle \tag{2.1}$$

$$N(m, \gamma) = \text{number of integers } k \text{ for which } 1 \leqq k \leqq m \text{ and } \langle kx \rangle \leqq \gamma, \tag{2.2}$$

and

$$t(\xi, \gamma) = \text{smallest integer } k \geqq 1 \text{ for which } \langle k\xi \rangle \leqq \gamma. \tag{2.3}$$

Clearly

$$Z(m) \leqq \alpha/m$$

is equivalent to

$$N(m, \alpha/m) \geqq 1 \tag{2.4}$$

which in turn is equivalent to

$$t(x, \alpha/m) \leqq m. \tag{2.5}$$

Hence, if the limits exist(3),

$$\begin{aligned} \lim_{m\to\infty} P\{Z(m) \leqq \alpha/m\} &= \lim_{m\to\infty} P\{N(m, \alpha/m) \geqq 1\} \\ &= \lim_{m\to\infty} P\{t(x, \alpha/m) \leqq m\}. \end{aligned} \tag{2.6}$$

(2) We shall always use x or x_i for random variables, whereas fixed numbers from $[0, 1]$ will be denoted by ξ or ξ_i.

(3) $P\{A\}$ = probability of the event A, $P\{A|B\}$ = conditional probability of the event A, given B, $E(X)$ = expectation of the random variable X, $E(X|B)$ = conditional expectation of the random variable X given B, $\text{var}(X) = E(X^2) - (EX)^2$ = variance of the random variable X, $\text{var}(X|B) = E(X^2|B) - (E(X|B))^2$ = conditional variance of the random variable X given B.

The last limit in (2.6) can be found by the methods of Friedman and Niven [3] and of Erdös, Szüsz and Turan [2]. In [3] the first two moments of $t(x, \alpha/m)$ were computed. Let F_k denote the Farey series of order k, that is the series of rational numbers(4) a/b, $0 \leqq a \leqq b \leqq k$, $(a, b) = 1$, in ascending order (cf. [5, Chapter III for more details]). It was proved in [3, p. 27] that for every ξ

$$t(\xi, 1/n) \leqq n - 1.$$

Writing further, for some fixed n,

$$I\left(\frac{a}{b}\right) = \left[\frac{a}{b} - \frac{1}{nb}, \frac{a}{b} + \frac{1}{nb}\right]$$

it was also shown in [3, p. 27] that

$$t(\xi, 1/n) \leqq k \tag{2.7}$$

if and only if one has

$$\xi \in I\left(\frac{a}{b}\right) \qquad \text{for some } \frac{a}{b} \in F_k. \tag{2.8}$$

Moreover, there are at most two fractions $a/b \in F_k$ $(k \leqq n-1)$ for which $\xi \in I(a/b)$ [2; 3]. Let

$$S_k = \left\{\xi: 0 \leqq \xi \leqq 1; \text{there exist } \frac{a_1}{b_1}, \frac{a_2}{b_2} \in F_k, \frac{a_1}{b_1} \neq \frac{a_2}{b_2} \text{ such that } \xi \in I\left(\frac{a_1}{b_1}\right) \text{ and } \xi \in I\left(\frac{a_2}{b_2}\right)\right\}.$$

Thus S_k is the subset of [0, 1] covered by two intervals I from F_k. If $\mu\{\cdot\}$ denotes Lebesgue measure, then it follows from the above remarks that (cf. [2, formula (12)])

$$\begin{aligned} P\left\{t\left(x, \frac{1}{n}\right) \leqq k\right\} &= \sum_{a/b \in F_k} \mu\left\{I\left(\frac{a}{b}\right)\right\} - \mu\{S_k\} = \sum_{a/b \in F_k} \frac{2}{nb} - \mu\{S_k\} \\ &= \frac{2}{n} \sum_{b=1}^{k} \frac{\Phi(b)}{b} - \mu\{S_k\} \end{aligned} \tag{2.9}$$

where $\Phi(\cdot)$ is Euler's function [5, p. 52].

It remains to find an expression for $\mu\{S_k\}$. If ξ is contained in two intervals $I(a_1/b_1)$, $I(a_2/b_2)$ with $a_i/b_i \in F_k$, $k \leqq n-1$, then [3, p. 27] a_1/b_1 and a_2/b_2 must be consecutive elements of F_k. It is known [5, Theorem 28, p. 23] that in this case

(4) $[a]$ = largest integer which does not exceed a. Although we also use square brackets with other meaning, confusion seems unlikely. (a, b) = greatest common divisor of a and b.

$$(2.10)\qquad \left|\frac{a_1}{b_1}-\frac{a_2}{b_2}\right|=\frac{|a_1b_2-a_2b_1|}{b_1b_2}=\frac{1}{b_1b_2}$$

and hence

$$(2.11)\qquad \mu\left\{I\left(\frac{a_1}{b_1}\right)\cap I\left(\frac{a_2}{b_2}\right)\right\}=\begin{cases}\dfrac{b_1+b_2-n}{nb_1b_2} & \text{if } b_1+b_2\geqq n,\\ 0 & \text{if } b_1+b_2<n.\end{cases}$$

One has therefore

$$(2.12)\qquad \mu\{S_k\}=\sum_k{}'\frac{b_1+b_2-n}{nb_1b_2}$$

where $\sum_k'$ ranges over all pairs a_1/b_1, a_2/b_2 which are consecutive elements of F_k and satisfy $b_1<b_2$ and $b_1+b_2\geqq n$. ($b_1=b_2$ is impossible for successive fractions by Theorem 28, p. 23 in [5] and since we want to take every pair into account *only once* we may take $b_1<b_2$.) It was proved in Lemma 1 of [3] that for fixed $b_1<b_2\leqq k$, $(b_1, b_2)=1$, $b_1+b_2\geqq n$ there are exactly two choices for a_1, a_2 such that a_1/b_1, a_2/b_2 belong to $\sum_k'$ while for $(b_1, b_2)>1$ there are no choices possible for a_1, a_2 (again by Theorem 28, p. 23 of [5]). Consequently

$$(2.13)\qquad \mu\{S_k\}=\sum_k{}'\frac{b_1+b_2-n}{nb_1b_2}=\frac{2}{n}\sum_{b_1=1}^{k}\sum_{\substack{b_2=\max(n-b_1,b_1+1)\\(b_1,b_2)=1}}^{k}\frac{b_1+b_2-n}{b_1b_2}.$$

With these preliminaries it is easy to prove

THEOREM 1.

$$\lim_{m\to\infty}P\left\{m\cdot\min_{1\leqq k\leqq m}\langle kx\rangle\leqq\alpha\right\}=\lim_{m\to\infty}P\left\{N\left(m,\frac{\alpha}{m}\right)\geqq 1\right\}$$
$$=\lim_{m\to\infty}P\left\{t\left(x,\frac{\alpha}{m}\right)\leqq m\right\}=F(\alpha)$$

where

$$(2.14)\qquad F(\alpha)=\begin{cases}0 & \textit{if } \alpha<0,\\ \dfrac{12\alpha}{\pi^2} & \textit{if } 0<\alpha\leqq 1/2,\\ \dfrac{12\alpha}{\pi^2}-\dfrac{12}{\pi^2}\displaystyle\int_{1/2}^{\alpha}\left(2-\frac{1}{y}-\frac{1-y}{y}\log\frac{y}{1-y}\right)dy & \textit{if } 1/2<\alpha\leqq 1,\\ 1 & \textit{if } 1<\alpha.\end{cases}$$

Proof. Firstly,

$$\lim_{k\to\infty} \frac{1}{k} \sum_{b=1}^{k} \frac{\Phi(b)}{b} = \frac{6}{\pi^2} \tag{2.15}$$ [3, p. 29].

Moreover, from (2.11) or (2.13) $\mu\{(S_k)\}=0$ for $k \leqq n/2$ because $b_1 < b_2 \leqq k$ implies $b_1+b_2<n$. For $k>n/2$ we have, following [3](4),

$$\begin{aligned}
&\frac{2}{n}\sum_{b_1=1}^{k} \sum_{\substack{b_2=\max(n-b_1,b_1+1)\\(b_1,b_2)=1}}^{k} \frac{b_1+b_2-n}{b_1b_2}\\
&= \frac{2}{n}\left(\sum_{b_1=n-k}^{[(n-1)/2]} \sum_{b_2=n-b_1}^{k} + \sum_{b_1=[(n+1)/2]}^{k} \sum_{b_2=b_1+1}^{k}\right)\frac{b_1+b_2-n}{b_1b_2} \sum_{d\mid(b_1,b_2)} \mu(d)\\
&= T_1 + T_2, \text{ say,}
\end{aligned} \tag{2.16}$$

where $\mu(\cdot)$ is the Möbius function [5, p. 234].

As an example we shall compute the asymptotic behaviour of T_1, the computation for T_2 being very similar. Changing the order of summation and putting $b_2=sd$ and

$$\{a\} = \text{smallest integer greater or equal to } a\text{(5)}$$

one has

$$T_1 = \frac{2}{n} \sum_{b=n-k}^{[(n-1)/2]} \frac{1}{b} \sum_{d\mid b} \mu(d) \sum_{s=\{(n-b)/d\}}^{[k/d]} \frac{b+sd-n}{sd}.$$

However,

$$\sum_{s=\{(n-b)/d\}}^{[k/d]} \frac{b+sd-n}{sd} = \frac{b-n}{d} \log \frac{k}{n-b} + \frac{k-n+b}{d} + O(1).$$

Since [5, p. 235 and p. 260]

$$\sum_{d\mid b} \frac{\mu(d)}{d} = \frac{\Phi(b)}{b}, \qquad \sum_{d\mid b} 1 = O(b^{1/2}),$$

$$\begin{aligned}
T_1 = {}& 2 \sum_{b=n-k}^{[(n-1)/2]} \frac{\Phi(b)}{b^2}\left(1-\frac{b}{n}\right)\log\left(1-\frac{b}{n}\right) + \frac{2k}{n} \sum_{b=n-k}^{[(n-1)/2]} \frac{\Phi(b)}{b^2}\\
&+ 2\left(1+\log\frac{k}{n}\right)\frac{1}{n} \sum_{b=n-k}^{[(n-1)/2]} \frac{\Phi(b)}{b} - 2\left(1+\log\frac{k}{n}\right) \sum_{b=n-k}^{[(n-1)/2]} \frac{\Phi(b)}{b^2}\\
&+ O\left(\frac{1}{n}+\frac{1}{k}\right).
\end{aligned}$$

Using the fact that [5, p. 268]

(5) This meaning of $\{a\}$ is used only in the next two formulae.

$$\sum_{b=1}^{n} \Phi(b) = \frac{3n^2}{\pi^2} + O(n \log n)$$

one obtains by partial summation as $n \to \infty$, $k \to \infty$, $k/n \to \alpha$, $1/2 \leqq \alpha < 1$,

$$\lim_{k/n \to \alpha} T_1 = \frac{12}{\pi^2}\left[\int_{1-\alpha}^{1/2} \frac{1-y}{y} \log(1-y)dy - \alpha \log 2(1-\alpha) \right. \tag{2.17}$$
$$\left. + (1+\log\alpha)(\alpha - 1/2 + \log 2(1-\alpha))\right].$$

In an entirely similar manner one obtains

$$\lim_{k/n \to \alpha} T_2 = \frac{12}{\pi^2}\left[\alpha \log 2\alpha - (\alpha - 1/2) - \int_{1/2}^{\alpha} \frac{y-1}{y} \log y dy \right. \tag{2.18}$$
$$\left. + \log \alpha \int_{1/2}^{\alpha} \frac{y-1}{y} dy\right].$$

(2.17) and (2.18) imply

$$\lim_{k/n \to \alpha} (T_1 + T_2) = \frac{12}{\pi^2}\int_{1/2}^{\alpha}\left(2 - \frac{1}{y} - \frac{1-y}{y}\log\frac{y}{1-y}\right)dy \tag{2.19}$$

as one easily checks by comparing the derivatives of the right-hand sides of (2.17)–(2.19) as well as the values at $\alpha = 1/2$.

From (2.9), (2.13), (2.15), (2.16), (2.19) one has

$$\lim_{k/n \to \alpha} P\left\{t\left(x, \frac{1}{n}\right) \leqq k\right\} = F(\alpha), \tag{2.20}$$

where $F(\alpha)$ is defined in (2.14). To prove the general relation

$$\lim_{m \to \infty} P\left\{t\left(x, \frac{\alpha}{m}\right) \leqq m\right\} = F(\alpha) \tag{2.21}$$

we notice, that if

$$\frac{1}{n+1} \leqq \frac{\alpha}{m} \leqq \frac{1}{n} \tag{2.22}$$

then for every ξ

$$t\left(\xi, \frac{1}{n}\right) \leqq t\left(\xi, \frac{\alpha}{m}\right) \leqq t\left(\xi, \frac{1}{n+1}\right).$$

Consequently also

$$P\left\{t\left(x, \frac{1}{n}\right) \leqq m\right\} \geqq P\left\{t\left(x, \frac{\alpha}{m}\right) \leqq m\right\} \geqq P\left\{t\left(x, \frac{1}{n+1}\right) \leqq m\right\}$$

which, together with (2.20), (2.22) completes the proof of (2.21).

Till now we considered integers $k \leqq m$ for which $\langle k\xi\rangle \leqq \alpha/m$. In [2] one considered integers k for which

$$\langle k\xi\rangle \leqq \frac{\alpha}{k} \cdot$$

In particular, putting ($\alpha > 0$, $c > 1$)

$$S(m, \alpha, c) = \{\xi\colon 0 \leqq \xi \leqq 1, \text{ there exist integers } a, b \text{ for which } m \leqq b \leqq mc, (a, b) = 1, |b\xi - a| \leqq \alpha/b\}.$$

Erdös, Szüsz and Turan [2] raised the question of finding

$$\lim_{m\to\infty} \mu\{S(m, \alpha, c)\}$$

if it exists at all. They found this limit for $\alpha \leqq c/(1+c^2)$ and gave bounds for $\mu\{S(m, \alpha, c)\}$ in several other cases. One has of course

$$\left\{\xi\colon 0 \leqq \xi \leqq 1, m \leqq t\left(\xi, \frac{\alpha}{mc}\right) \leqq mc\right\} \subseteq S(m, \alpha, c)$$

because

$$\left|t\left(\xi, \frac{\alpha}{mc}\right)\xi - r\right| \leqq \frac{\alpha}{mc} \leqq \frac{\alpha}{t(\xi, \alpha/mc)} \tag{2.23}$$

if $t(\xi, \alpha/mc) \leqq mc$, and at the same time (2.23) implies $(t(\xi, \alpha/mc), r) = 1$ (cf. [3, p. 27]).

Consequently one has

$$\liminf_{m\to\infty} \mu\{S(m, \alpha, c)\} \geqq F(\alpha) - F\left(\frac{\alpha}{c}\right). \tag{2.24}$$

Entirely obvious is the inclusion

$$S(m, \alpha, c) \subseteq \left\{\xi\colon 0 \leqq \xi \leqq 1, t\left(\xi, \frac{\alpha}{m}\right) \leqq mc\right\},$$

and therefore

$$\limsup_{m\to\infty} \mu\{S(m, \alpha, c)\} \leqq F(\alpha c). \tag{2.25}$$

(2.24) and (2.25) are improvements on the results of [2] for certain combinations of α and c. (2.25) however is only useful if $\alpha c < 1$ (compare (2.14)).

In this case, however, we can compute the limit of $\mu\{S(m, \alpha, c)\}$ exactly. The result is given by the next theorem.

THEOREM 2. *If* $\alpha \leqq c/(1+c^2)$, *then*

$$\lim_{m\to\infty} \mu\{S(m, \alpha, c)\} = \frac{12\alpha}{\pi^2} \log c \qquad [2]. \tag{2.26}$$

If $c/(1+c^2) \leqq \alpha \leqq \min(1/2, 1/c)$, *then*

$$\lim_{m\to\infty} \mu\{S(m, \alpha, c)\} = \frac{12\alpha}{\pi^2}\log c - \frac{12}{\pi^2}\left(\alpha c + \frac{\alpha}{c} - \alpha\beta - \frac{\alpha}{\beta} + \alpha\left(\frac{1}{\beta} - \beta\right)\log\frac{c}{\beta} - \frac{1}{2}\left(\log\frac{c}{\beta}\right)^2\right), \tag{2.27}$$

where

$$\beta = \frac{1 + (1 - 4\alpha^2)^{1/2}}{2\alpha}.$$

If $1/2 \leqq \alpha \leqq 1/c$, *then*

$$\lim_{m\to\infty} \mu\{S(m, \alpha, c)\} = \frac{12\alpha}{\pi^2}\log c - \frac{12}{\pi^2}\left(\alpha c - 2\alpha + \frac{\alpha}{c} - \frac{1}{2}(\log c)^2\right). \tag{2.28}$$

Added in proof. Since this paper was written the following two references containing results related to this theorem have come to the author's attention: P. Erdös, *Some results on diophantine approximation*, Acta Arithmetica **5** (1959), 359–369 and Richard P. Gosselin, *On diophantine approximation and trigonometric polynomials*, Pacific J. Math. **9** (1959), 1071–1081.

Proof. (2.26) is Theorem III of [2]. Instead of $I(a/b)$ we now define

$$J\left(\frac{a}{b}\right) = \left[\frac{a}{b} - \frac{\alpha}{b^2}, \frac{a}{b} + \frac{\alpha}{b^2}\right] \qquad \text{for } \frac{a}{b} \in F_{[mc]}.$$

Then

$$S(m, \alpha, c) = \bigcup_{a/b \in F_{[mc]};\, b \geq m} J\left(\frac{a}{b}\right). \tag{2.29}$$

$$\sum_{a/b\in F_{[mc]};\, b\geq m} \mu\left\{J\left(\frac{a}{b}\right)\right\} = \sum_{m}^{[mc]} \frac{2\alpha\Phi(b)}{b^2} \to \frac{12\alpha}{\pi^2}\log c \qquad (m\to\infty). \tag{2.30}$$

In addition, if a_1/b_1 and a_2/b_2 are two consecutive elements of $F_{[mc]}$ with $b_1, b_2 \geqq m$, then

$$\left|\frac{a_2}{b_2}-\frac{a_1}{b_1}\right|=\frac{1}{b_1b_2}\geqq \max\left(\frac{\alpha}{b_1^2},\frac{\alpha}{b_2^2}\right)$$

by [5, Theorem 28, p. 23] and the fact that $b_1/b_2\leqq c$, $b_2/b_1\leqq c$, $\alpha c\leqq 1$. Consequently $a_1/b_1\notin J(a_2/b_2)$, $a_2/b_2\notin J(a_1/b_1)$ and therefore no ξ can be in more than two intervals $J(a/b)$, $b\geqq m$, $a/b\in F_{[mc]}$. If ξ is in $J(a_1/b_1)\cap J(a_2/b_2)$ then a_1/b_1 and a_2/b_2 must be consecutive elements of $F_{[mc]}$. In this case

$$\mu\left\{J\left(\frac{a_1}{b_1}\right)\cap J\left(\frac{a_2}{b_2}\right)\right\} = \max\left(0,\frac{\alpha}{b_1^2}+\frac{\alpha}{b_2^2}-\frac{1}{b_1b_2}\right).$$

Notice that for $\alpha\geqq 1/2$, always

$$\frac{\alpha}{b_1^2}+\frac{\alpha}{b_2^2}-\frac{1}{b_1b_2}\geqq 0$$

where as for $\alpha\leqq 1/2$, $b_2>b_1$

$$\frac{\alpha}{b_1^2}+\frac{\alpha}{b_2^2}-\frac{1}{b_1b_2}\geqq 0$$

only if

$$\frac{b_2}{b_1}\geqq\beta=\frac{1+(1-4\alpha^2)^{1/2}}{2\alpha}\cdot$$

Thus, by (2.29) and (2.30)

$$\mu\{S(m,\alpha,c)\} = \frac{12\alpha}{\pi^2}\log c-\sum{}''\left(\frac{\alpha}{b_1^2}+\frac{\alpha}{b_2^2}-\frac{1}{b_1b_2}\right)+o(1)$$

where $\sum''$ ranges over all pairs a_1/b_1, a_2/b_2 of consecutive elements of $F_{[mc]}$ with

(2.31) $$m\leqq b_1<b_2\leqq mc \qquad \text{if } \alpha\geqq 1/2$$

and with

(2.32) $$m\leqq b_1,\qquad \beta b_1\leqq b_2\leqq mc\qquad \text{if } \alpha\leqq 1/2.$$

It follows again from Lemma 1 in [3], that for given b_1, b_2 satisfying (2.31) if $\alpha\geqq 1/2$ or (2.32) if $\alpha\leqq 1/2$, there are exactly two pairs (a_1, a_2) such that a_1/b_1, a_2/b_2 belong to $\sum''$ if $(b_1,b_2)=1$ and no such pairs if $(b_1,b_2)>1$. $\sum''$ can now be computed exactly as in Theorem 1.

3. **The distribution of $N(m,\gamma)$ in more dimensions.** As we have seen in the last section, the study of the distribution of $\min_{1\leq k\leq m}\langle kx\rangle$ was equivalent to finding $P\{N(m,\gamma)=0\}$ for appropriate γ. This raises the question of finding the complete distribution of $N(m,\gamma)$. Even though the methods of §2 probably allow us to find the asymptotic distribution of $N(m,\alpha/m)$, it seems

very hard to find the asymptotic distribution of $N(m,\gamma)-EN(m,\gamma)=N(m,\gamma)-2m\gamma$ after proper normalization, for fixed γ. The difficulty seems to be the "strong" dependence between the random variables $\langle kx\rangle$, $k=1, 2, \cdots$. This shows a.o. in the fact that the distribution of $m^{-1/2}(N(m,\gamma)-2m\gamma)$ does *not* approach a normal distribution, as it would if the random variables $\langle kx\rangle$ were strictly independent. In fact $m^{-1/2}$ is not at all the correct normalization factor [7]. It was suggested by M. Kac in a discussion with the author that independence would approximately hold again for analogous random variables in high dimensions. This will be shown to be correct in a certain sense in the next theorem. The limiting distribution obtained in Theorem 3 is precisely the limiting distribution which would pertain if the Y's were strictly independent.

Let $x_1, x_2, \cdots$ be independent random variables, each with a uniform distribution on $[0, 1]$. Define

$$Y_k^j(\gamma) = \begin{cases} 1 & \text{if } \langle kx_1\rangle \leqq \gamma,\ \langle kx_2\rangle \leqq \gamma, \cdots, \langle kx_j\rangle \leqq \gamma, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$N(m, \gamma, j) = \sum_{k=1}^{m} Y_k^j(\gamma).$$

$N(m, \gamma, j)$ is the number of indices $k \leqq m$ for which *simultaneously* $kx_1, \cdots, kx_j$ are closer than γ to an integer.

THEOREM 3. *If* $0<\gamma<1/2$ *is fixed and* $p\to\infty$, $m\to\infty$ *such that*

$$m(2\gamma)^p \to \lambda > 0, \tag{3.1}$$

then

$$\lim P\{N(m, \gamma, p) = k\} = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, \cdots,$$

that is, $N(m, \gamma, p)$ *has asymptotically a Poisson distribution with mean* λ.

Proof. Put for $k_1, \cdots, k_r$ pairwise different

$$A_\gamma(k_1, \cdots, k_r) = P\{Y_{k_1}^1(\gamma) = \cdots = Y_{k_r}^1(\gamma) = 1\}. \tag{3.2}$$

γ will be fixed throughout $(0<\gamma<1/2)$ and the index or subscript γ will only be written explicitly when necessary to avoid confusion. Since the $x_1, x_2, \cdots$ are independent one has clearly

$$\begin{aligned} P\{Y_{k_1}^j = \cdots = Y_{k_r}^j = 1\} &= (P\{Y_{k_1}^1 = \cdots = Y_{k_r}^1 = 1\})^j \\ &= A^j(k_1, \cdots, k_r). \end{aligned} \tag{3.3}$$

Moreover,

$$\begin{aligned}(3.4)\quad E(N(m,\gamma,j)(N(m,\gamma,j)-1)(N(m,\gamma,j)-2)\cdots(N(m,\gamma,j)-r+1))\\ = E\sum_{k(r)} Y^j_{k_1}\cdots Y^j_{k_r} = \sum_{k(r)} A^j(k(r)).\end{aligned}$$

We adopted here the convention to write $k(r)$ for a generic r-tuple $(k_1, \cdots, k_r)$ of different integers k_i, $1 \leqq k_i \leqq m$, and to include in $\sum_{k(r)}$ all such r-tuples in the $r!$ orders in which they can appear. This convention will be used through the remainder of this section.

One easily checks that if N has a Poisson distribution with mean λ, then its rth factorial moment,

$$E(N(N-1)(N-1)\cdots(N-r+1)) = \lambda^r, \qquad r = 1, 2, \cdots.$$

Conversely, if $\{N(p)\}$ is a sequence of random variables for which

$$\lim_{p\to\infty} E(N(p)(N(p)-1)(N(p)-2)\cdots(N(p)-r+1)) = \lambda^r, \quad r = 1, 2, \cdots$$

then the limiting distribution $(p\to\infty)$ of $N(p)$ is a Poisson distribution with mean λ (by [9, p. 185 C; 6, p. 115 4.30, p. 109 4.21]).

Theorem 3 will therefore be proved, if we can show

$$\lim \sum_{k(r)} A^p(k(r)) = \lambda^r, \qquad r = 1, 2, \cdots,$$

or equivalently, by (3.1)

$$(3.5)\qquad \lim \frac{\sum_{k(r)} \left(\frac{A_\gamma(k(r))}{(2\gamma)^r}\right)^p}{m(m-1)\cdots(m-r+1)} = 1, \qquad r = 1, 2, \cdots.$$

We shall prove (3.5) by induction on r.

Let us put

$$(3.6)\qquad \nu_r^{(j)}(\gamma) = \frac{\sum_{k(r)} \left(\frac{A_\gamma(k(r))}{(2\gamma)^r}\right)^j}{m(m-1)\cdots(m-r+1)}.$$

Then

$$\nu_1^{(p)}(\gamma) = \frac{\sum_{k=1}^m A^p_\gamma(k)}{m(2\gamma)^p} = \frac{\sum_{k=1}^m (P\{Y^1_k(\gamma)=1\})^p}{m\cdot(2\gamma)^p} = 1$$

since

$$P\{Y^1_k(\gamma) = 1\} = P\{\langle kx_1\rangle \leqq \gamma\} = 2\gamma.$$

Thus (3.5) holds for $r=1$. The remaining details will be split up in a number of lemmas. C_i, $i=1, 2, \cdots$, will always stand for some finite, positive constant which depends on γ, r, λ only.

LEMMA 1. *There exists a C_1 such that for any sequence of s positive integers $k_1<k_2<\cdots<k_s$*

$$E\left(\sum_{j=1}^{s} Y_{k_j}^1(\gamma) - 2s\gamma\right)^2 \leqq \frac{s}{2}\,(C_1 + (\log s)^2).$$

C_1 does not depend on s.

Proof[3].

$$\begin{aligned} E\left(\sum_{j=1}^{s} Y_{k_j}^1 - 2s\gamma\right)^2 &= \operatorname{var}\left(\sum_{j=1}^{s} Y_{k_j}^1\right) \\ &= s2\gamma(1-2\gamma) + 2\sum_{1\leq i<j\leq s} (A(k_i, k_j) - 4\gamma^2). \end{aligned} \tag{3.7}$$

It was proved in [8, p. 217 line 4] that

$$\left| A_\gamma(k_i, k_j) - 4\gamma^2 \right| \leqq \frac{(k_i, k_j)^2}{k_i\cdot k_j} = \frac{1}{k_i'\cdot k_j'} \tag{3.8}$$

where (k_i, k_j) = greatest common divisor of k_i and k_j and

$$k_i' = \frac{k_i}{(k_i, k_j)}, \qquad k_j' = \frac{k_j}{(k_i, k_j)}.$$

Notice that for any pair of integers $a<b$ there are at most s possible pairs $k_i<k_j$ such that $k_i'=a$, $k_j'=b$ and hence, for any positive integer c there are at most $sd(c)/2$ pairs $k_i<k_j$ with $k_i'\cdot k_j'=c$ ($d(c)=$ number of divisors of c). Therefore,

$$\sum_{1\leq i<j\leq s} \left| A_j(k_i, k_j) - 4\gamma^2 \right| \leqq \frac{s}{2}\sum_{c=1}^{c_0} \frac{d(c)}{c} \tag{3.9}$$

where c_0 is the smallest integer for which

$$\frac{s}{2}\sum_{c=1}^{c_0} d(c) \geqq \frac{s(s-1)}{2}.$$

In fact there are $s(s-1)/2$ pairs $k_i<k_j$ and we have replaced each $1/k_i'k_j'$ by some $1/c$ with $1/k_i'k_j' \leqq 1/c$. Since

$$\sum_{c=1}^{n} d(c) = n\log n + O(n)$$

[5, Theorem 320, p. 264],

$$\frac{s}{2}\sum_{c=1}^{c_0}\frac{d(c)}{c} \leqq \frac{s}{2}\sum_{c=1}^{2s/\log s}\frac{d(c)}{c} \leqq \frac{s}{4}(\log s)^2$$

for sufficiently large s. This together with (3.7) and (3.8) implies the lemma.

The estimate of [8] for $|A(k_i, k_j)-4\gamma^2|$ would suffice to prove directly $\nu_2^{(p)}(\gamma)\to 1$, and one might want to follow LeVeque's method [8] of estimating $A_\gamma(k_1, k_j)$ also for $A_\gamma(k_1, \cdots, k_r)$ in order to prove (3.5) directly for all r. This would require an estimate of

$$A\gamma(k_1, \cdots, k_r) = \int_0^1 d\xi \prod_{t=1}^{r}\left(2\gamma + \frac{2}{\pi}\sum_{n_t=1}^{\infty}\frac{\sin 2\pi n_t\gamma \cos 2\pi n_t k_t \xi}{n_t}\right).$$

We have been unable to follow this direct procedure and are forced to prove (3.5) by a detour.

Lemma 2. *If X is any random variable satisfying*

$$E(X-a)^2 \leqq c^2 \tag{3.10}$$

and

$$0 \leqq X \leqq b \quad \textit{with probability one,} \tag{3.11}$$

then, there exists constants C_2, independent of the distribution of X such that

$$|EX - a| \leqq c, \tag{3.12}$$

$$\int_{2a}^{\infty} y^r dP\{X \leqq y\} \leqq C_2(r)b^{r-2}c^2, \qquad r = 3, 4, \cdots, \tag{3.13}$$

and

$$|EX^r - (EX)^r| \leqq C_2(r)b^{r-2}c^2, \qquad r = 2, 3, 4, \cdots. \tag{3.14}$$

Proof. Put $EX=\mu$. Then

$$\begin{aligned} c^2 &\geqq E(X-a)^2 = E(X-\mu+\mu-a)^2 = E(X-\mu)^2 + (\mu-a)^2 \\ &\geqq (EX-a)^2. \end{aligned} \tag{3.15}$$

This proves (3.12). As for (3.13) and (3.14) we use the one sided analogue of Tchebyshev's inequality. For $\lambda \geqq 0$,

$$\begin{aligned} P\{X - a \geqq \lambda\} &\leqq \frac{1}{(\lambda^2+c^2)^2}\int_a^{\infty}(\lambda(y-a)+c^2)^2 dP\{X \leqq y\} \\ &\leqq \frac{2}{(\lambda^2+c^2)^2}\int_a^{\infty}(\lambda^2(y-a)^2+c^4)dP\{X \leqq y\} \\ &\leqq \frac{2}{(\lambda^2+c^2)^2}(\lambda^2c^2+c^4) = \frac{2c^2}{\lambda^2+c^2}\cdot \end{aligned} \tag{3.16}$$

Putting

$$G(\lambda) = \begin{cases} 0 & \text{if } \lambda \leqq c, \\ 1 - \dfrac{2c^2}{\lambda^2 + c^2} & \text{if } c \leqq \lambda < b - a, \\ 1 & \text{if } b - a \leqq \lambda, \end{cases}$$

one has by (3.11) and (3.16)

$$1 - P\{X \leqq a + \lambda\} \leqq 1 - G(\lambda).$$

Consequently, for $r \geqq 3$

$$\begin{aligned} \int_{2a}^{\infty} y^r dP\{X \leqq y\} &= \int_{2a}^{b+} y^r dP\{X \leqq y\} \\ &= -(1 - P\{X \leqq y\})y^r \Big|_{2a}^{b} + r\int_{2a}^{b} y^{r-1}(1 - P\{X \leqq y\})dy \\ &= (1 - P\{X \leqq 2a\})(2a)^r + r\int_{2a}^{b} y^{r-1}(1 - P\{X \leqq y\})dy \\ &\leqq (2a)^r(1 - G(a)) + r\int_{a}^{b-a} (z + a)^{r-1}(1 - G(z))dz \\ &\leqq (2a)^r \frac{2c^2}{a^2} + 2^{r-1}r\int_{a}^{b-a} z^{r-1}\cdot\frac{2c^2}{z^2 + c^2}\, dz. \end{aligned}$$

This implies (3.13) when $2a \leqq b$. When $2a > b$ (3.13) becomes trivial by (3.11).

(3.14) is proved similar to (3.13). By (3.11), $0 \leqq \mu \leqq b$ and by (3.15) $E(X-\mu)^2 \leqq c^2$. Thus, with a replaced by μ one has from (3.16)

$$1 - P\{X \leqq \mu + \lambda\} \leqq 1 - G(\lambda) \tag{3.17}$$

and similarly with a replaced by $b-\mu$

$$P\{X \leqq \mu - \lambda\} \leqq 1 - G(\lambda). \tag{3.18}$$

Now, since $E(X-\mu)$ equals zero,

$$\begin{aligned} EX^r = E(\mu + X - \mu)^r = \mu^r + \binom{r}{2}\mu^{r-2}E(X - \mu)^2 \\ + \sum_{j=3}^{r}\binom{r}{j}\mu^{r-j}E(X - \mu)^j. \end{aligned} \tag{3.19}$$

By (3.17), for $j \geqq 3$

$$\int_{y \geqq \mu} (y - \mu)^j dP\{X \leqq y\} \leqq \int_0^{b-\mu} z^j dG(z) \leqq C_3(j)b^{j-2}c^2 \tag{3.20}$$

and by (3.18)

$$\int_{y\leqq\mu} |y-\mu|^j dP\{X \leqq y\} \leqq C_3(j)b^{j-2}c^2. \tag{3.21}$$

(3.14) follows from (3.19)–(3.21).

An important consequence of Lemma 1 and Lemma 2 is the following:

LEMMA 3. *If $p\to\infty$, $m\to\infty$ such that $m(2\gamma)^p\to\lambda$, then for some C_4, depending on γ, r, λ only, and $j\leqq p$*

$$\exp(-C_4(2\gamma)^{p-j}(1+(p-j)^2)) \leqq \nu_r^{(j)}(\gamma) \leqq \exp(C_4(2\gamma)^{p-j}(1+(p-j)^2)). \tag{3.22}$$

Proof. As we remarked already, Lemma 3 is obvious for $r=1$, and we may assume $r\geqq 2$.

$$\begin{aligned}\nu_r^{(j)}(\gamma) &= \frac{E(N(m,\gamma,j)\cdots(N(m,\gamma,j)-r+1))}{m(m-1)\cdots(m-r+1)(2\gamma)^{rj}} \\ &\leqq \frac{EN^r(m,\gamma,j)}{m(m-1)\cdots(m-r+1)(2\gamma)^{rj}}.\end{aligned}$$

Denote by $F(k_1,\cdots,k_s;u)$ the event

$$Y_{k_1}^u = Y_{k_2}^u = \cdots = Y_{k_s}^u = 1 \quad \text{whereas} \quad Y_k^u = 0 \quad \text{if } k\neq k_i, \quad i=1,2,\cdots,s.$$

Then, dropping the indices m and γ for this proof,

$$EN^r(j) = \sum_{1\leq k_1<k_2\cdots<k_s\leq m} P\{F(k_1,\cdots,k_s;j-1)\}E(N^r(j)\mid F(k_1,\cdots,k_s;j-1)).$$

However,

$$E(N^r(j)\mid F(k_1,\cdots,k_s;j-1)) = E\left(\sum_{t=1}^{s} Y_{k_t}^1\right)^r \tag{3.23}$$

because if $Y_{k_1}^{j-1}=1$, then $Y_{k_1}^j=1$ if and only if $\langle k_1x_j\rangle\leqq\gamma$ and, if $Y_{k_1}^{j-1}=0$ then $Y_{k_1}^j$ always equals zero. Hence,

$$\begin{aligned}&E(N(j)\mid F(k_1,\cdots,k_s;j-1)) \leqq 2\gamma s,\\ &\operatorname{var}(N(j)\mid F(k_1,\cdots,k_s;j-1)) \leqq \frac{s}{2}(C_1+(\log s)^2) \qquad \text{(Lemma 1)}\end{aligned} \tag{3.24}$$

and since $0\leqq N(j)\leqq s$ if $F(k_1,\cdots,k_s;j-1)$ occurs,

$$E(N^r(j)\mid F(k_1,\cdots,k_s);j-1)) \leqq (2\gamma)^r s^r + C_2(r)\frac{s^{r-1}}{2}(C_1+(\log s)^2) \ \text{(by(3.14))}.$$

If we take into account that

$$N(j-1)=s \qquad \text{if } F(k_1, \cdots, k_s; j-1)$$

occurs one has

$$EN^r(j) \leqq \sum_{1 \leq k_1 < \cdots < k_s \leq m} P\{F(k_1, \cdots, k_s; j-1\}$$

$$\cdot\left[(2\gamma)^r N^r(j-1) + \frac{C_2}{2} N^{r-1}(j-1)(C_1 + (\log N(j-1))^2)\right] \tag{3.25}$$

$$= (2\gamma)^r EN^r(j-1) + \frac{C_2 C_1}{2} EN^{r-1}(j-1)$$

$$+ \frac{C_2}{2} EN^{r-1}(j-1)(\log N(j-1))^2.$$

Applying Jensen's inequality [9, p. 156, c,] twice gives

$$EN^{r-1}(j-1) \leqq (EN^r(j-1))^{(r-1)/r} \leqq \frac{EN^r(j-1)}{EN(j-1)} = \frac{EN^r(j-1)}{m\cdot(2\gamma)^{j-1}}\,. \tag{3.26}$$

The last term of (3.25) is estimated first by using Hölder's inequality [9, p. 156]

$$EN^{r-1}(j-1)(\log N(j-1))^2$$

$$\leqq (2r-1)^2 EN^{r-1}(j-1) + \int_{\exp(2r-1)}^{\infty} y^{r-1}(\log y)^2 dP\{N(j-1) \leqq y\}$$

$$\leqq (2r-1)^2 EN^{r-1}(j-1) + \left(\int_{\exp(2r-1)}^{\infty} y^r dP\{N(j-1) \leqq y\}\right)^{(r-1)/r}$$

$$\cdot\left(\int_{\exp(2r-1)}^{\infty} (\log y)^{2r} dP\{N(j-1) \leqq y\}\right)^{1/r}.$$

Since $d^2(\log y)^{2r}/dy^2 \leqq 0$ for $y \geqq \exp(2r-1)$, Jensen's inequality [4, Theorem 95, p. 77; 9, p. 159, e] implies

$$\int_{\exp(2r-1)}^{\infty} (\log y)^{2r} dP\{N(j-1) \leqq y\}$$

$$\leqq P\{N(j-1) \geqq \exp(2r-1)\} \left\{\log \frac{\int_{\exp(2r-1)}^{\infty} y\,dP\{N(j-1) \leqq y\}}{\int_{\exp(2r-1)}^{\infty} dP\{N(j-1) \leqq y\}}\right\}^{2r} \tag{3.27}$$

$$= O(\log EN(j-1))^{2r}$$

$$= O(\log m(2\gamma)^j)^{2r}.$$

Applying again (3.26) to $EN^{r-1}(j-1)$ and combining (3.25)–(3.27) one obtains for some $C_5(\gamma, r)$

$$EN^r(j) \leqq (2\gamma)^r EN^r(j-1)\left(1 + C_5 \frac{1 + (\log m(2\gamma)^j)^2}{m\cdot(2\gamma)^j}\right)$$

$$\leqq (2\gamma)^r EN^r(j-1) \exp\left(C_5 \frac{1 + (\log m(2\gamma)^j)^2}{m(2\gamma)^j}\right).$$

This also holds for $j=1$ if we put $N(0)=m$ with probability one. Thus by induction on j

$$EN^r(j) \leqq (2\gamma)^{rj} m^r \exp\left(C_5 \sum_{n=1}^{j} \frac{1 + (\log m(2\gamma)^n)^2}{m(2\gamma)^n}\right),$$

which immediately gives the right-hand inequality of (3.22) if we take into account $m(2\gamma)^p \to \lambda$.

In the same way one shows(6)

$$EN^r(j) \geqq (2\gamma)^{rj} m^r \exp\left(-C_5 \sum_{n=1}^{j} \frac{1 + (\log m(2\gamma)^n)^2}{m(2\gamma)^n}\right)$$

which implies the left-hand inequality of (3.22) because

$$EN(N-1)\cdots(N-r+1) = EN^r + O\left(E \sum_{k=0}^{r-1} N^k\right).$$

LEMMA 4. *There exists a constant $C_6(r, \gamma) < \infty$ such that*

$$A_\gamma^{-1}(k(r)) \leqq C_6(r, \gamma). \tag{3.28}$$

Proof.

$$2\gamma + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2\pi n\gamma \cos 2\pi n\xi}{n} = \begin{cases} 1 & \text{if } \langle\xi\rangle < \gamma, \\ 0 & \text{if } \langle\xi\rangle > \gamma \end{cases}$$

(cf. [8]). Hence

$$A_\gamma(k_1, \cdots, k_r) = \int_0^1 d\xi \prod_{t=1}^{r} \left(2\gamma + \frac{2}{\pi} \sum_{n_t=1}^{\infty} \frac{\sin 2\pi n_t\gamma \cos 2\pi n_t k_t \xi}{n_t}\right).$$

Since

(6) We shall use the left-hand inequality of (3.22) only for $j \leqq (2p+6)/3$. For this range the argument in the next few lines suffices but not for all $j \leqq p$. However, the left-hand inequality of (3.22) is true for all $j \leqq p$. The same remark applies to the left-hand inequality of (3.45).

$$\sum_{n=M+1}^{\infty} \frac{\sin 2\pi n\gamma \cos 2\pi nk\xi}{n}$$

is bounded uniformly in M, k, ξ (being the tail of the Fourier series of a function of bounded variation [10, p. 408]) and since

$$\int_0^1 \left| \sum_{n=M+1}^{\infty} \frac{\sin 2\pi n\gamma \cos 2\pi nk\xi}{n} \right| d\xi = \int_0^1 \left| \sum_{n=M+1}^{\infty} \frac{\sin 2\pi n\gamma \cos 2\pi n\xi}{n} \right| d\xi$$

tends to zero as $M \to \infty$ (uniformly in k), it is possible to choose $M = M(\gamma, t)$ such that

$$\begin{aligned} &A(k_1, \cdots, k_r) \\ &\geqq \int_0^1 d\xi \prod_{t=1}^{r} \left(2\gamma + \frac{2}{\pi} \sum_{n_t=1}^{M} \frac{\sin 2\pi n_t\gamma \cos 2\pi n_t k_t \xi}{n_t} \right) - \frac{(2\gamma)^r}{2} \\ &= \frac{(2\gamma)^r}{2} + \sum_{j=1}^{r} (2\gamma)^{r-j} \left(\frac{2}{\pi} \right)^j \sum_{1 \leq t_i \leq r} \int_0^1 d\xi \prod_{i=1}^{j} \frac{\sin 2\pi n_{t_i}\gamma \cos 2\pi n_{t_i} k_{t_i} \xi}{n_{t_i}}. \end{aligned} \tag{3.29}$$

The last integrals can only be different from zero if some relation

$$\sum_{t=1}^{r} \epsilon_t n_t k_t = 0, \qquad \epsilon_t = 0, +1, -1$$

holds, with at least two ϵ's different from zero. We can now prove the lemma by induction on r. $A_\gamma(k_1) = 2\gamma$ i.e. $C_6(1, \gamma) = (2\gamma)^{-1}$ satisfies (3.28) for $r = 1$. Let (3.28) be proved already for $r-1$ and assume there exists a sequence $(k_1^{(n)}, \cdots, k_r^{(n)})$ $n = 1, 2, \cdots$ of r-tuples $k^{(n)}(r)$ such that

$$\lim_{n \to \infty} A(k^{(n)}(r)) = 0. \tag{3.30}$$

By virtue of (3.29) and (3.30) we may assume, if necessary by selecting a subsequence and rearranging the indices, that for some fixed $\epsilon_1, \cdots, \epsilon_r$ and $n_1, \cdots, n_r \leqq M(\gamma, r)$ and all n

$$\sum_{t=1}^{r} \epsilon_t n_t k_t^{(n)} = 0, \qquad \epsilon_{r-1}\epsilon_r \neq 0.$$

In this case, however, one has for any ξ satisfying

$$\langle k_j^{(n)} \xi \rangle \leqq \frac{\gamma}{rM}, \qquad j = 1, \cdots, r-1,$$

also

$$\langle n_r k_j^{(n)} \xi \rangle \leqq \frac{\gamma}{r} \leqq \gamma, \qquad j = 1, \cdots, r-1$$

and

$$\langle n_r k_r^{(n)} \xi \rangle = \left\langle \sum_{t=1}^{r-1} \epsilon_t n_t k_t^{(n)} \xi \right\rangle \leqq \frac{(r-1)M}{rM} \gamma \leqq \gamma.$$

Consequently,

$$\begin{aligned} A_\gamma(n_r k_1^{(n)}, n_r k_2^{(n)}, \cdots, n_r k_r^{(n)}) &\geqq A_{\gamma/rM}(k_1^{(n)}, \cdots, k_{r-1}^{(n)}) \\ &\geqq C_6\left(r-1, \frac{\gamma}{rM}\right)^{-1}. \end{aligned} \tag{3.31}$$

But $A(k_1^{(n)}, \cdots, k_r^{(n)}) = A(n_r k_1^{(n)}, \cdots, n_r k_r^{(n)})$ because $n_r x$ has a uniform distribution modulo one if x has. (3.31) contradicts (3.30) which proves the lemma.

Lemma 5. *There exists a positive constant $C_7(r, \gamma)$ for every $r \geqq 2$, such that for every $k(r) = (k_1, \cdots, k_r)$*

$$\frac{A_\gamma(k(r))}{A_\gamma(k_1, \cdots, k_{j-1}, k_{j+1}, \cdots, k_r)} \leqq 1 - C_7(r, \gamma) < 1$$

for some $1 \leqq j \leqq r$.

Proof. For shortness put $k(r, j) = (k_1, \cdots, k_{j-1}, k_{j+1}, \cdots, k_r)$. By the definition of A

$$\begin{aligned} \frac{A(k(r))}{A(k(r,j))} &= P\{ Y_{k_j}^1 = 1 \mid Y_{k_i}^1 = 1, 1 \leqq i \leqq r, i \neq j\} \\ &= 1 - A^{-1}(k(r,j)) P\{ Y_{k_j}^1 = 0, Y_{k_i}^1 = 1, 1 \leqq i \leqq r, i \neq j\}. \end{aligned} \tag{3.32}$$

It therefore suffices to prove

$$\begin{aligned} B(k(r,j)) &= P\{ Y_{k_j}^1 = 0, Y_{k_i}^1 = 1, 1 \leqq i \leqq r, i \neq j\} \\ &\geqq C_8(r,j) > 0 \qquad \text{for some } 1 \leqq j \leqq r \end{aligned} \tag{3.33}$$

where $C_8(r, j)$ does not depend on $k(r)$.

Assume (3.33) does not hold and that $k^{(n)}(r)$, $n = 1, 2, \cdots$, is a sequence of r-tuples for which

$$\lim_{n \to \infty} B(k^{(n)}(r, j)) = 0, \qquad j = 1, \cdots, r. \tag{3.34}$$

Without loss of generality we may rearrange the indices and select a subsequence such that

$$k_1^{(n)} < k_2^{(n)} < \cdots < k_r^{(n)} \qquad \text{for all } n \tag{3.35}$$

and

$$\lim_{n\to\infty} \frac{k_r^{(n)}}{k_i^{(n)}} \geqq 1 \qquad \text{exists for } i = 1, \cdots, r \tag{3.36}$$

(infinity is allowed as a limit in (3.36)). Let

$$\lim_{n\to\infty} \frac{k_r^{(n)}}{k_i^{(n)}} = 1 \qquad \text{for } i = s+1, \cdots, r \tag{3.37}$$

while

$$\lim_{n\to\infty} \frac{k_r^{(n)}}{k_i^{(n)}} > 1 \qquad \text{for } i = 1, \cdots, s.$$

By deleting some n's we can assume

$$\frac{k_j^{(n)}}{k_i^{(n)}} \geqq 1 + \delta, \tag{3.38}$$

$i = 1, \cdots, s, \quad j = s+1, \cdots, r$ for all n and some $0 < \delta \leqq 1$.

Let $g(\xi)$ be the fractional part of ξ minus the integer closest to ξ. More precisely

$$g(\xi) = \begin{cases} \xi & \text{if } 0 \leqq \xi \leqq 1/2, \\ \xi - 1 & \text{if } 1/2 < \xi \leqq 1, \end{cases}$$

$$g(\xi + 1) = g(\xi).$$

One has

$$P\{ | g(k_i x) - g(k_j x) | \leqq \eta\} \leqq P\{ | g((k_i - k_j)x) | \leqq \eta\} \leqq 2\eta$$

so that for

$$\eta = \min\left[\left(4\binom{r}{2} C_6\left(r, \frac{\gamma\delta}{4(1+\delta)}\right)\right)^{-1}, \frac{1}{\gamma} - 2, \frac{\delta}{(2+\delta)}\right], \tag{3.39}$$

$$\begin{aligned} P&\left\{ | g(k_i x) | \leqq \frac{\gamma\delta}{4(1+\delta)}, | g(k_i x) - g(k_j x) | \geqq \eta, 1 \leqq i,j \leqq r, i \neq j\right\} \\ &\geqq A_{\gamma\delta(4+4\delta)^{-1}}(k(r)) - \sum_{1 \leqq i < j \leqq r} P\{ | g(k_i x) - g(k_j x) | \leqq \eta\} \\ &\geqq C_6^{-1}\left(r, \frac{\gamma\delta}{4(1+\delta)}\right) - \binom{r}{2} \cdot 2\eta \\ &\geqq \frac{1}{2} C_6^{-1}\left(r, \frac{\gamma\delta}{4(1+\delta)}\right) > 0. \end{aligned} \tag{3.40}$$

There exists therefore a ξ, depending on n and satisfying

$$(3.41)\qquad |g(k_i^{(n)}\xi)| \leqq \frac{\gamma\delta}{4(1+\delta)}, \quad |g(k_i^{(n)}\xi) - g(k_j^{(n)}\xi)| \geqq \eta,$$

$1 \leqq i, j \leqq r$, $i \neq j$. Let j_1 be determined by

$$g(k_{j_1}^{(n)}\xi) = \max_{s+1\leqq j\leqq r} g(k_j^{(n)}\xi) \qquad (\text{thus } j_1 \geqq s+1)$$

and put

$$(3.42)\qquad \bar{\xi} = \frac{\gamma(1+\eta/2) - g(k_{j_1}^{(n)}\xi)}{k_{j_1}^{(n)}} \leqq \frac{\gamma(1+\eta/2+\delta/4(1+\delta))}{k_{j_1}^{(n)}}.$$

Then

$$(3.43)\qquad g(k_{j_1}^{(n)}\xi) + k_{j_1}^{(n)}\bar{\xi} = \gamma\left(1+\frac{\eta}{2}\right) \leqq \frac{1}{2}.$$

On the other hand, by (3.38), (3.41) and (3.42), for any $i \leqq s$

$$g(k_i^{(n)}\xi) + k_i^{(n)}\bar{\xi} \leqq \frac{\gamma\delta}{4(1+\delta)} + \frac{\gamma(1+\eta/2+\delta/4(1+\delta))}{1+\delta} \leqq \gamma\left(1-\frac{\eta}{2}\right),$$

and for any $i \geqq s+1$, $i \neq j_1$

$$(3.44)\qquad \begin{aligned} g(k_i^{(n)}\xi) + k_i^{(n)}\bar{\xi} &\leqq g(k_{j_1}^{(n)}\xi) - \eta + k_{j_1}^{(n)}\bar{\xi} + (k_i^{(n)} - k_{j_1}^{(n)})\bar{\xi} \\ &\leqq \gamma\left(1+\frac{\eta}{2}\right) - \eta + \frac{|k_i^{(n)} - k_{j_1}^{(n)}|}{k_{j_1}^{(n)}}\gamma\left(1+\frac{\eta}{2}+\frac{\delta}{4(1+\delta)}\right). \end{aligned}$$

Since, for $i, j_1 \geqq s+1$

$$\frac{k_i^{(n)}}{k_{j_1}^{(n)}} \to 1$$

the last member of (3.44) will eventually also be less than $\gamma(1-\eta/2)$. Thus for sufficiently large n

$$g(k_{j_1}^{(n)}\xi) + k_{j_1}^{(n)}\bar{\xi} = g(k_{j_1}^{(n)}(\xi+\bar{\xi})) = \gamma\left(1+\frac{\eta}{2}\right)$$

and by (3.41)–(3.44)

$$-\frac{\gamma\delta}{4(1+\delta)} \leqq g(k_i^{(n)}(\xi+\bar{\xi})) \leqq \gamma\left(1-\frac{\eta}{2}\right) \qquad \text{if } i \neq j_1.$$

If now

$$\langle k_i^{(n)} t\rangle < \gamma \frac{\eta}{2}, \qquad i = 1, \cdots, r,$$

then

$$\langle k_i^{(n)}(\xi + \bar{\xi} + t)\rangle \leqq \gamma\left(1 - \frac{\eta}{2} + \frac{\eta}{2}\right) = \gamma, \qquad i \neq j_1,$$

whereas for j_1

$$\langle k_{j_1}^{(n)}(\xi + \bar{\xi} + t)\rangle \geqq \left| g(k_{j_1}^{(n)}(\xi + \bar{\xi}))\right| - \langle k_{j_1}^{(n)} t\rangle > \gamma.$$

Thus, for sufficiently large n,

$$B(k^{(n)}(r, j_1)) \geqq \mu\left\{v: v = \xi + \bar{\xi} + t, 0 \leqq t \leqq 1, \langle k_i^{(n)} t\rangle < \gamma \frac{\eta}{2}, i = 1, \cdots, r\right\}$$

$$= P\left\{\langle k_i^{(n)} x\rangle < \gamma \frac{\eta}{2}, i = 1, \cdots, r\right\} = A_{\gamma\eta/2}(k^{(n)}(r)) \geqq C_6^{-1}\left(r, \frac{\gamma\eta}{2}\right).$$

This contradicts (3.34) and therefore proves the lemma.

LEMMA 6. *If $p \to \infty$, $m \to \infty$ such that $m(2\gamma)^p \to \lambda$ then for some C_9, depending on γ, r, λ only, and $j \leqq p$*

$$\exp(-C_9(2\gamma)^{(p-j)/2}(1 + p - j))$$

$$\leqq \frac{1}{m - r + 1} \sum_{\substack{k \neq k_1, \cdots, k_{r-1} \\ 1 \leq k \leq m}} \left(\frac{A_\gamma(k_1, \cdots, k_{r-1}, k)}{A_\gamma(k_1, \cdots, k_{r-1}) 2\gamma}\right)^j \tag{3.45}$$

$$\leqq \exp(C_9(2\gamma)^{(p-j)/2}(1 + p - j)).$$

Proof. Put

$$\nu^{(j)}(\gamma, k(r-1)) = \frac{1}{m - r + 1} \sum_{\substack{k \neq k_1, \cdots, k_{r-1} \\ 1 \leq k \leq m}} \left(\frac{A_\gamma(k_1, \cdots, k_{r-1}, k)}{A_\gamma(k_1, \cdots, k_{r-1}) 2\gamma}\right)^j. \tag{3.46}$$

In this proof we assume $k(r-1)$ and γ fixed and shall abbreviate $\nu^{(j)}(\gamma, k(r-1))$ by $\nu^{(j)}$. For the same fixed $k(r-1)$ and γ we put

$$N'(j) = \sum_{\substack{k \neq k_1, \cdots, k_r \\ 1 \leq k \leq m}} Y_k^j(\gamma).$$

Then (cf. (3.32)),

$$\nu^{(j)} = \frac{E(N'(j) \mid Y_{k_1}^j = \cdots = Y_{k_{r-1}}^j = 1)}{(m - r + 1)\cdot(2\gamma)^j}.$$

The proof will very much resemble the proof of Lemma 3.

$$
\begin{aligned}
&E(N'(j) \mid Y^j_{k_1} = \cdots = Y^j_{k_{r-1}} = 1) \\
(3.47) \quad &= \sum_{\substack{1 \leq t_1 < \cdots < t_s \leq m \\ t_u \neq k_v}} P\{F(k_1, \cdots, k_{r-1}, t_1, \cdots, t_s; j-1) \mid Y^{j-1}_{k_1} = \cdots = Y^{j-1}_{k_{r-1}} = 1\} \\
&\quad \cdot E(N'(j) \mid Y^j_{k_1} = \cdots = Y^j_{k_{r-1}} = 1, F(k_1, \cdots, k_{r-1}, t_1, \cdots, t_s; j-1)).
\end{aligned}
$$

However,

$$E(N'(j) \mid F(k_1, \cdots, k_{r-1}, t_1, \cdots, t_s; j-1)) = 2\gamma s$$

and thus by Lemma 1 (compare (3.23), (3.24)),

$$E((N'(j) - 2\gamma s)^2 \mid F(k_1, \cdots, k_{r-1}, t_1, \cdots, t_s; j-1)) \leq \frac{s}{2}(C_1 + (\log s)^2).$$

Hence, by Lemma 4

$$
\begin{aligned}
&E((N'(j) - 2\gamma s)^2 \mid Y^{(j)}_{k_1} = \cdots = Y^{(j)}_{k_{r-1}} = 1, \\
&\qquad\qquad F(k_1, \cdots, k_{r-1}, t_1, \cdots t_s; j-1)) \\
(3.48) \quad &\leq \frac{s(C_1 + (\log s)^2)}{2P\{Y^{(j)}_{k_1} = \cdots = Y^{(j)}_{k_{r-1}} = 1 \mid F(k_1, \cdots, k_{r-1}, t_1, \cdots, t_s; j-1)} \\
&= \frac{s(C_1 + (\log s)^2)}{2A(k_1, \cdots, k_{r-1})} \leq C_6(r-1, \gamma)\frac{s}{2}(C_1 + (\log s)^2).
\end{aligned}
$$

By (3.12) and (3.48) one has

$$
\begin{aligned}
&E(N'(j) \mid Y^j_{k_1} = \cdots = Y^j_{k_{r-1}} = 1, \\
(3.49) \quad &\qquad F(k_1, \cdots, k_{r-1}, t_1, \cdots, t_s; j-1)) - 2\gamma s) \\
&\qquad \leq C_{10}(r, \gamma)s^{1/2}(1 + \log s)
\end{aligned}
$$

for appropriate C_{10}.

Since, under the condition $F(k_1, \cdots, k_{r-1}, t_1, \cdots, t_s; j-1)$

$$N'(j-1) = s,$$

(3.47) and (3.49) give immediately

$$
\begin{aligned}
&\mid E(N'(j) \mid Y^j_{k_1} = \cdots \\
&\quad = Y^j_{k_{r-1}} = 1) - 2\gamma E(N'(j-1) \mid Y^{j-1}_{k_1} = \cdots = Y^{j-1}_{k_{r-1}} = 1) \mid \\
(3.50) \quad &\leq C_{10}E(N'^{1/2}(j-1)(1 + \log N'(j-1)) \mid Y^{j-1}_{k_1} = \cdots = Y^{j-1}_{k_{r-1}} = 1) \\
&\leq 5C_{10}E(N'^{3/4}(j-1) \mid Y^{j-1}_{k_1} = \cdots = Y^{j-1}_{k_{r-1}} = 1).
\end{aligned}
$$

Thus

$$(3.51)\qquad \nu^{(j)} \leqq \nu^{(j-1)} + \frac{5C_{10}(\nu^{(j-1)})^{3/4}}{(m-r+1)^{1/4}(2\gamma)^{j/4+1}}, \qquad \nu^{(0)} = 1.$$

First we get from (3.51) by induction,

$$\nu^{(j)} \leqq \exp\left(5C_{10}\sum_{u=0}^{j-1}(m-r+1)^{-1/4}(2\gamma)^{-u/4-1}\right)$$

which is bounded for $j \leqq p$. Then we complete the proof from (3.50) as in Lemma 3 by applying Hölder's and Jensen's inequalities and induction on j(6).

An immediate corollary of (3.45) is

$$(3.52)\qquad \frac{1}{m-r+1}\sum_{\substack{k\neq k_1,\cdots,k_{r-1}\\ 1\leq k\leq m}}\left(\frac{A_\gamma(k_1,\cdots,k_{r-1},k)}{A_\gamma(k_1,\cdots,k_{r-1})2\gamma}\right)^{3[p/3]+3} \leqq (2\gamma)^{-3}\exp C_9 = C_{11}, \text{ say.}$$

We are now in the position to prove (3.5). Put

$$U_r(k(r)) = \left(\frac{A(k(r))}{(2\gamma)^r}\right)^{[p/3]+1}.$$

If we consider U_r as a random variable taking each of the values $U_r(k(r))$ with probability $1/m(m-1)\cdots(m-r+1)$ then (3.6) and Lemma 3 state

$$EU_r = \frac{1}{m(m-1)\cdots(m-r+1)}\sum_{k(r)}\left(\frac{A(k(r))}{(2\gamma)^r}\right)^{[p/3]+1}$$
$$\geqq \exp(-C_4(2\gamma)^{(2p-3)/3}p^2) \quad \text{for sufficiently large } p.$$

Similarly

$$EU_r^2 \leqq \exp C_4(2\gamma)^{(p-6)/3}p^2$$

and consequently

$$\operatorname{var}(U_r) = EU_r^2 - (EU_r)^2 \leqq C_{12}(\gamma, r, \lambda)p^2(2\gamma)^{p/3}$$

for appropriate C_{12}. In other words

$$U_r \to 1 \quad \text{in probability} \quad (p \to \infty)$$

and

$$\frac{1}{m(m-1)\cdots(m-r+1)}\sum_{k(r)}\left(\frac{A(k(r))}{(2\gamma)^r}\right)^p = EU_r^{p/([p/3]+1)}$$

will indeed tend to one if we can show

$$(3.53)\qquad \lim \frac{1}{m(m-1)\cdots(m-r+1)}\sum_{k(r);U^3_r\geq s^r} U_r^3(k(r)) = 0$$

[9, p. 184 B].

Unfortunately, Lemma 3 alone does not seem to be strong enough to prove (3.53), and we have to proceed by induction. (3.53) certainly holds for $r=1$ and let us assume it has already been proved with r replaced by $r-1$. We shall then prove that it also holds for r. For this purpose, we define

$$(3.54)\quad V_r(k(r)) = \begin{cases} \left(\dfrac{A_\gamma(k_1, \cdots, k_r)}{A_\gamma(k_1, \cdots, k_{r-1})2\gamma}\right)^{[p/3]+1} & \text{if } \dfrac{A_\gamma(k_1, \cdots, k_r)}{A_\gamma(k_1, \cdots, k_{r-1})} \leqq C_{13}(\gamma, r), \\ \dfrac{1}{m-r+1} \displaystyle\sum_{k \neq k_1, \cdots, k_{r-1}} \left(\dfrac{A_\gamma(k_1, \cdots, k_{r-1}, k)}{A_\gamma(k_1, \cdots, k_{r-1})2\gamma}\right)^{[p/3]+1} & \text{otherwise} \end{cases}$$

where

$$(3.55)\qquad C_{13}(\gamma, r) = \max(1 - C_7(\gamma, r), (2\gamma)) < 1 \qquad \text{(cf. Lemma 5).}$$

Any set of r different integers $k_1, \cdots, k_r \leqq m$ will appear in $r!$ orders as a $k(r)$. For some j

$$\frac{A_\gamma(k(r))}{A_\gamma(k(r, j))} \leqq C_{13}(\gamma, r) \qquad \text{by Lemma 5.}$$

This j will appear in $(r-1)!$ permutations of $1, \cdots, r$ at the end. Thus taking into account

$$(3.56)\quad \begin{gathered} \left(\frac{A(k(r))}{(2\gamma)^r}\right)^{3[p/3]+3} = \left(\frac{A(k(r-1))}{(2\gamma)^{r-1}}\right)^{3[p/3]+3} \left(\frac{A(k(r))}{A(k(r-1))2\gamma}\right)^{3[p/3]+3}, \\ \sum_{\substack{k(r) \\ U_r \geq 3^r}} U_r^3(k(r)) \leqq r \sum_{k(r-1)} U_{r-1}^3(k(r-1)) \sum_{\substack{k_r \neq k_1, \cdots, k_{r-1} \\ U_{r-1}^3 V_r^3 \geq 3^r}} V_r^3(k(r)). \end{gathered}$$

Since $U^3 V^3 \geqq 3^r$ implies $U^3 \geqq 3^{r-1}$ or $V \geqq 3$ one obtains from (3.56)

$$(3.57)\quad \begin{aligned} &\frac{1}{m(m-1)\cdots(m-r+1)} \sum_{\substack{k(r) \\ U_r \geq 3^r}} U_r^3(k(r)) \\ &\leqq \frac{r}{m(m-1)\cdots(m-r+2)} \sum_{\substack{k(r) \\ U_{r-1} \geq 3^{r-1}}} U_{r-1}^3(k(r-1)) \frac{1}{m-r+1} \\ &\qquad \cdot \sum_{k_r \neq k_1, \cdots, k_{r-1}} V_r^3(k(r)) \\ &\quad + \frac{r}{m(m-1)\cdots(m-r+2)} \sum_{k(r-1)} U_{r-1}^3(k(r-1)) \frac{1}{m-r+1} \\ &\qquad \cdot \sum_{\substack{k_r \neq_1 k, \cdots, k_{r-1} \\ V_r \geq 3}} V_r^3(k(r)). \end{aligned}$$

By the induction hypothesis and (3.52) the first sum in the right-hand side of (3.57) tends to zero. As for the second term, we have

$$\begin{aligned} &\frac{1}{m(m-1)\cdots(m-r+2)}\sum_{k(r-1)} U_{r-1}^2(k(r-1))\frac{1}{m-r+1} \\ (3.58)\quad &\cdot\sum_{k\neq k_1,\cdots,k_{r-1}}\left(\frac{A(k_1,\cdots,k_{r-1},k)}{A(k(r-1))2\gamma}\right)^{2[p/3]+2} \leqq \exp(C_4(2\gamma)^{(p-6)/3}p^2\end{aligned}$$

by (3.6) and Lemma 3. On the other hand by (3.6), Lemma 3 and Lemma 6

$$\begin{aligned} &\frac{1}{m(m-1)\cdots(m-r+2)}\sum_{k(r-1)} U_{r-1}^2(k(r-1)) \\ (3.59)\quad &\cdot\left(\frac{1}{m-r+1}\sum_{k\neq k_1,\cdots,k_{r-1}}\left(\frac{A(k_1,\cdots,k_{r-1},k)}{A(k(r-1))2\gamma}\right)^{[p/3]+1}\right)^2 \\ &\geqq \exp(-C_4(2\gamma)^{(p-6)/3}p^2)\exp(-2C_9(2\gamma)^{(p-3)/3}p).\end{aligned}$$

Putting

$$D(k(r-1)) = \frac{1}{m-r+1}\sum_{\substack{k\neq k_1,\cdots,k_{r-1}\\ 1\leq k\leq m}}\left(\frac{A(k_1,\cdots,k_{r-1},k)}{A(k(r-1))2\gamma}\right)^{[p/3]+1}$$

and subtracting (3.59) from (3.58) one sees

$$\begin{aligned} &\frac{1}{m(m-1)\cdots(m-r+2)}\sum_{k(r-1)} U_{r-1}^2(k(r-1)) \\ (3.60)\quad &\cdot\frac{1}{m-r+1}\sum_{k\neq k_1,\cdots,k_{r-1}}\left(\left(\frac{A(k_1,\cdots,k_{r-1},k)}{A(k(r-1))2\gamma}\right)^{[p/3]+1} - D(k(r-1))\right)^2 \\ &\leqq C_{14}(\gamma,r,\lambda)(2\gamma)^{p/3}p^2\end{aligned}$$

for appropriate C_{14}. Thus

$$\begin{aligned} &\frac{1}{m-r+1}\sum_{k\neq k_1,\cdots,k_{r-1}}\left(\left(\frac{A(k_1,\cdots,k_{r-1},k)}{A(k(r-1))2\gamma}\right)^{[p/3]+1} - D(k(r-1))\right)^2 \\ (3.61)\quad &\leqq C_{14}(2\gamma)^{p/3}p^2C_{13}^{-p/6}\end{aligned}$$

except for a set S of $(r-1)$-tuples $k(r-1)$ for which

$$(3.62)\qquad \frac{1}{m(m-1)\cdots(m-r+2)}\sum_{k(r-1)\in S} U_{r-1}^2(k(r-1)) \leqq C_{13}^{p/6}.$$

However, by (3.52)

$$\frac{1}{m(m-1)\cdots(m-r+2)}\sum_{k(r-1)\in S} U_{r-1}^{3}(k(r-1))\,\frac{1}{m-r+1}\sum_{k\neq k_1,\cdots,k_{r-1}} V_r^{3}(k(r))$$
$$\leqq \frac{2C_{11}}{m(m-1)\cdots(m-r+2)}\sum_{k(r-1)\in S} U_{r-1}^{3}(k(r-1))$$
$$\leqq \frac{2C_{11}}{m(m-1)\cdots(m-r+2)}\sum_{\substack{k(r-1)\\ U_{r-1}\geqq 3^{r-1}}} U_{r-1}^{3}(k(r-1))$$
$$+\frac{2C_{11}\cdot 3^{r-1}}{m(m-1)\cdots(m-r+2)}\sum_{k(r-1)\in S} U_{r-1}^{2}(k(r-1)).$$

Both these terms tend to zero, the first by the induction hypothesis and the second by (3.62) and (3.55). In view of (3.57) it remains to prove

$$\tag{3.63} \frac{1}{m(m-1)\cdots(m-r+2)}\sum_{k(r-1)\notin S} U_{r-1}^{3}(k(r-1)) \cdot\frac{1}{m-r+1}\sum_{\substack{k_r\neq k_1,\cdots,k_{r-1}\\ V_r\geqq 3}} V_r^{3}(k(r)) \to 0.$$

Let $k(r-1)\notin S$ be fixed and look at V_r as a random variable which takes each of the values $V_r(k(r))$ with probability $1/(m-r+1)$. Then

$$E(V_r - D_r(k(r-1)))^2$$
$$\leqq \frac{1}{m-r+1}\sum_{k\neq k_1,\cdots,k_{r-1}}\left(\left(\frac{A(k_1,\cdots,k_{r-1},k)}{A(k(r-1))2\gamma}\right)^{[p/3]+1} - D(k(r-1))\right)^2$$
$$\leqq C_{14}(2\gamma)^{p/3}p^2C_{13}^{-p/6} \qquad \text{by (3.61).}$$

On the other hand

$$0\leqq V_r \leqq \max\left(\left(\frac{C_{13}}{2\gamma}\right)^{[p/3]+1},\qquad D(k(r-1))\right)$$

while $D(k(r-1))\leqq 3/2$ for sufficiently large p by Lemma 6. Thus, by Lemma 2, (3.13)

$$\tag{3.64} \frac{1}{m-r+1}\sum_{\substack{k\neq k_1,\cdots,k_r\\ V_r\geqq 3}} V_r^{3}(k(r)) \leqq C_2(3)\left(\frac{3}{2}+\left(\frac{C_{13}}{2\gamma}\right)^{[p/3]+1}\right)\cdot C_{14}(2\gamma)^{p/3}p^2C_{13}^{-p/6}\to 0 \qquad (p\to\infty).$$

Since, by Lemma 3

$$\frac{1}{m(m-1)\cdots(m-r+2)}\sum_{k(r-1)} U_{r-1}^{3}(k(r-1))$$

is uniformly bounded, (3.63) follows from (3.64) and this implies (3.53). By induction (3.53) and (3.5) follow now for all r and this proves the theorem.

COROLLARY. *If $p\to\infty$, $m\to\infty$ such that*

$$m(2\gamma)^p \to \infty \qquad \textit{sufficiently slow,}$$

then

$$P\left\{\frac{N(m,\gamma,p)-m(2\gamma)^p}{(m(2\gamma)^p)^{1/2}} \leqq \alpha\right\} \to \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\alpha} e^{-t^2/2}dt.$$

Proof. This follows from Theorem 3, since for random variables $X(\lambda)$, having a Poisson distribution with mean λ, one has [6, p. 116, Ex. 4.9]

$$\lim_{\lambda\to\infty} P\left\{\frac{X(\lambda)-\lambda}{\lambda^{1/2}} \leqq \alpha\right\} = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\alpha} e^{-t^2/2}dt.$$

Our estimates are not sharp enough, however, to tell how fast $m(2\gamma)^p$ may tend to infinity.

One might want to prove Theorem 3 for any interval of length 2γ. More precisely, put

$$Y_k^j(\delta, 2\gamma+\delta) = \begin{cases} 1 & \text{if there exist integers } n_1, \cdots, n_j \text{ such that} \\ & \qquad \delta + n_i \leqq kx_i \leqq \delta + 2\gamma + n_i,\ i = 1, \cdots, j, \\ 0 & \text{otherwise.} \end{cases}$$

As long as

$$-\frac{1}{2} < \delta < 0 < \delta + 2\gamma < \frac{1}{2} \tag{3.65}$$

one can still follow the proof of Theorem 3. In fact, the only places in the above proof where a difference between $Y(\gamma)$ and $Y(\delta, 2\gamma+\delta)$ might come in are the Lemmas 1, 4 and 5. However, the estimate (3.8) remains valid (cf. [8]) so that the proof of Lemma 1 needs no change and one easily sees also that Lemmas 4 and 5 remain valid. This is not so when (3.65) is violated. The analogue of Lemma 4 is then false for certain choices of γ and δ. We formulate this with another generalization in

THEOREM 4. *If γ, δ satisfy* (3.65) *and if $p\to\infty$, $m\to\infty$ such that $m(2\gamma)^p\to\lambda$ and if for every m, $k_1(m), \cdots, k_m(m)$ are m different positive integers, then*

$$\lim P\left\{\sum_{i=1}^{m} Y_{k_i(m)}^{p}(\delta, 2\gamma+\delta) = k\right\} = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \cdots.$$

The transition to arbitrary sequences $\{k_i(m),\ i=1, \cdots, m\}$ of integers requires no change of proof since our estimates are uniform in $\{k_i(m)\}$. Finally we mention the possibility of taking intervals of length 2γ at a random location. I.e. let $\delta_1, \delta_2, \cdots$ also be random variables, independent of each other and of $x_1, x_2, \cdots$, and each with a uniform distribution on $[0, 1]$. Put

$$\tilde{Y}_k^j(\gamma) = \begin{cases} 1 & \text{if there exist integers } n_1, \cdots, n_j \text{ such that} \\ & \qquad \delta_i + n_i \leqq kx_i \leqq \delta_i + 2\gamma + n_i,\ i = 1, \cdots, j, \\ 0 & \text{otherwise.} \end{cases}$$

Then Theorem 4 holds with $Y(\delta, 2\gamma+\delta)$ replaced by $\tilde{Y}(\gamma)$. Again the proof is almost identical with that of Theorem 3.

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